Chapter 9

CURVE SKETCHING

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at that point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise then that the Cartesian coordinate system in which we graph functions, and the calculus that we use to analyze functions were invented in close succession, in the seventeenth century. In this chapter, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

CHAPTER EXPECTATIONS In this chapter, you will

- compare rates of change of graphs of functions, **Section 9.1**
- determine properties of the graphs of polynomial functions, **Section 9.1, 9.5**
- describe key features of a given graph of a function, **Section 9.1, 9.2, 9.4**
- determine intercepts and positions of the asymptotes to a graph, **Section 9.3**
- sketch the graph of a function, **Section 9.4**
- determine key features of the graph of a function, **Section 9.5, Career Link**
- sketch, by hand, the graph of the derivative of a given graph, **Section 9.2**
- determine from the equation of a simple combination of polynomial, rational, or exponential functions $\left(\text{e.g., } f(x) = x^2 + \frac{1}{x}\right)$ the key features of the graph of the combination of functions, using the techniques of differential calculus, and sketch the graph by hand, **Section 9.4**

Review of Prerequisite Skills

When we are sketching the graph of a function, there are many features that we can analyze in order to help us create the sketch. For example, we can try to determine the x- and y-intercepts for the graph, we can test for horizontal and vertical asymptotes using limits, and we can use our knowledge of certain kinds of functions to help us determine ranges, domains, and possible symmetries.

In this chapter, we will use what we have learned about the derivatives of functions, in conjunction with all the things mentioned above, to learn more about functions and their graphs. In approaching these concepts, you should first

- be able to solve simple equations and inequalities.
- know how to sketch graphs of basic functions and simple transformations of these graphs (including parabolas, and logarithmic and exponential functions).
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits.
- be able to find the derivatives of functions using all known rules.
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits.

Exercise

1. In the following, solve the given equation.

   a. $2y^2 + y - 3 = 0$
   b. $x^2 - 5x + 3 = 17$
   c. $4x^2 + 20x + 25 = 0$
   d. $y^3 + 4y^2 + y - 6 = 0$

2. In the following, solve the given inequality.

   a. $3x + 9 < 2$
   b. $5(3 - x) \geq 3x - 1$
   c. $t^2 - 2t < 3$
   d. $x^2 + 3x - 4 > 0$

3. In the following, sketch the graph of the given function.

   a. $f(x) = (x + 1)^2 - 3$
   b. $f(x) = x^2 - 5x - 6$
   c. $f(x) = 1 - 2^x$
   d. $f(x) = \log_{10}(x + 4)$

4. In the following, evaluate the given limits.

   a. $\lim_{x \to 2^-} (x^2 - 4)$

   b. $\lim_{x \to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

   c. $\lim_{x \to 0} x^2 3^{-x}$

   d. $\lim_{x \to 2} \log_5(x - 1)$

5. In the following, find the derivative of the given function.

   a. $f(x) = \dfrac{1}{4}x^4 + 2x^3 - \dfrac{1}{x}$

   b. $f(x) = \dfrac{x + 1}{x^2 - 3}$

   c. $f(x) = e^{-x^2}$

   d. $f(x) = x^5 \ln(x)$

6. Divide each of the following and write your answer in the form $ax + b + \dfrac{r}{q(x)}$. For example, $(x^2 + 4x - 5) \div (x - 2) = x + 6 + \dfrac{7}{x - 2}$.

   a. $(x^2 - 5x + 4) \div (x + 3)$

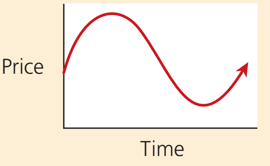
   b. $(x^2 + 6x - 9) \div (x - 1)$

Career Link investigate

CHAPTER 9: PREDICTING STOCK VALUES

Stock-market analysts collect and interpret vast amounts of information and then predict trends in stock values. Stock analysts are broken down into two main groups, the fundamentalists who predict stock values based on analysis of the companies' economic situations, and the technical analysts who predict stock values based on trends and patterns in the market. Technical analysts spend a significant amount of their time constructing and interpreting graphical models that are used to find undervalued stocks that will give returns in excess of what the market predicts. In this chapter, your skills in producing and analyzing graphical models will be extended through the use of differential calculus.

Case Study: Technical Stock Analyst



In order to raise money to expand operations, many privately owned companies give the public a chance to own a part of their company through purchasing stock. Those who buy ownership anticipate obtaining a share in future profits of the company. Some technical analysts believe that the greatest profits to be had in the stock market are through buying brand new stocks and selling them quickly. A technical analyst predicts that a stock's price over its first several weeks on the market will follow the pattern shown on the graph. The technical analyst is advising a person who purchased the stock the day it went on sale.

DISCUSSION QUESTIONS

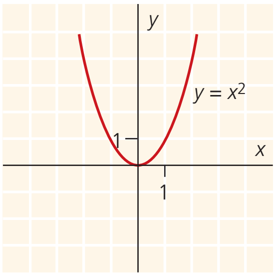
Make a rough sketch of the graph and answer the following questions:

1. When would you recommend the owner sell her shares? Mark this point on your graph with an "S." What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
2. When would you recommend the owner get back into the company and buy shares again? Mark this point on your graph with a "B." What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
3. A concave-down section of a graph is one that opens down, and similarly, concave up opens up. Mark a "C" on the graph when the concavity changes from concave down to concave up. A fellow analyst says that a change in concavity from concave down to concave up is a signal that a selling opportunity will soon occur. Do you agree with your fellow analyst? Explain.

At the end of this chapter, you will have an opportunity to apply the tools of graph sketching to create, evaluate, and apply a model that will be used to provide advice to clients on when to buy, sell, and hold new stocks.

Section 9.1 — Increasing and Decreasing Functions

The graph of the function $f(x) = x^2$ is a parabola. If we imagine a particle moving along this parabola from left to right, we can see that while the x-coordinates of the ordered pairs steadily increase, the y-coordinates of the ordered pairs along the particle's path first decrease then increase. Determining intervals in which a function increases or decreases is of great use in understanding the behaviour of the function. The following statements give a clear picture:

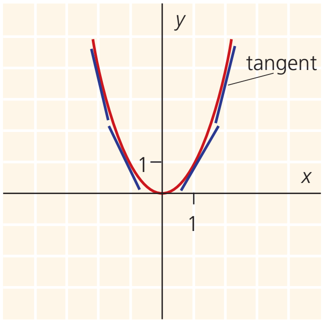


We say that a function, f, *is decreasing on an interval* if, for any $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.

Similarly, we say that a function, f, *is increasing on an interval* if, for any $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.

For the parabola with equation $y = x^2$, the change from decreasing y-values to increasing y-values occurs at the vertex of the parabola, which is (0, 0). The function $f(x) = x^2$ is decreasing on the interval $x < 0$ and is increasing on the interval $x > 0$.

If we examine a line tangent to the parabola anywhere on the interval where the y-values are decreasing (i.e., on $x < 0$), we see that all of these tangents have negative slopes. Similarly, the slopes of lines tangent to the increasing portion of the graph are all positive.



For functions that are both continuous and differentiable, we can determine intervals of increasing and decreasing y-values using the derivative of the function. In the case of $y = x^2$, $\frac{dy}{dx} = 2x$. For $x < 0$, $\frac{dy}{dx} < 0$, and the slopes of the tangents are negative. The interval $x < 0$ corresponds to the decreasing portion of the graph of

the parabola. For $x > 0$, $\frac{dy}{dx} > 0$ and the slopes of the tangents are positive on the interval where the graph is increasing.

We summarize this as follows:

For a continuous and differentiable function f, the function values (y-values) are increasing for all x-values where $f'(x) > 0$, and the function values (y-values) are decreasing for all x-values where $f'(x) < 0$.

EXAMPLE

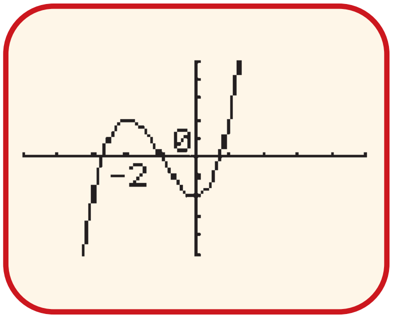
Use your calculator to obtain graphs of the following functions. Use the graph to estimate the values of x for which the function (y-values) is increasing, and for which values of x the function is decreasing. Verify your estimates with an analytic solution.

a. $y = x^3 + 3x^2 - 2$ b. $y = x^2e^{-x}$

Solution

technology

a. Using a calculator, we obtain the graph of $y = x^3 + 3x^2 - 2$. Using the TRACE key on the calculator, we estimate that the function values are increasing on $x < -2$, decreasing on $-2 < x < 0$, and increasing again on $x > 0$. To verify these estimates with an analytic solution, we consider the slopes of the tangents.



The slope of a general tangent to the graph of $y = x^3 + 3x^2 - 2$ is given by $\frac{dy}{dx} = 3x^2 + 6x$. We first determine values of x for which $\frac{dy}{dx} = 0$. These values tell us where the function has a maximum or minimum value.

Setting $\frac{dy}{dx} = 0$, we obtain

$$3x^2 + 6x = 0$$
$$3x(x + 2) = 0$$
$$x = 0, x = -2$$

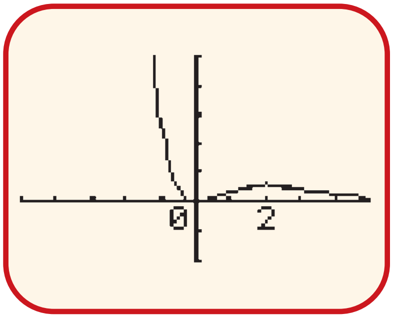
These values of x locate points on the graph at which the slope of the tangent is zero (i.e., horizontal).

Since $\frac{dy}{dx}$ is defined for all values of x, and since $\frac{dy}{dx} = 0$ only at $x = -2$ and $x = 0$, it must be either positive or negative for all other values of x. We consider the intervals $x < -2$, $-2 < x < 0$, and $x > 0$.

Value of x	Value of $\frac{dy}{dx} = 3x(x + 2)$	Slope of Tangent	y-Values Increasing or Decreasing
$x < -2$	$\frac{dy}{dx} > 0$	positive	increasing
$-2 < x < 0$	$\frac{dy}{dx} < 0$	negative	decreasing
$x > 0$	$\frac{dy}{dx} > 0$	positive	increasing

Then $y = x^3 + 3x^2 - 2$ is increasing on the intervals $x < -2$ and $x > 0$ and is decreasing on the interval $-2 < x < 0$.

b. Using a calculator, we obtain the graph of $y = x^2e^{-x}$. Using the TRACE key on the calculator, we estimate that the function values (y-values) are decreasing on $x < 0$, increasing on $0 < x < 2$, and decreasing again on $x > 2$.



We analyze the intervals of increasing/decreasing y-values for the function by determining where $\frac{dy}{dx}$ is positive and where it is negative.

$$\begin{aligned}\frac{dy}{dx} &= 2xe^{-x} + x^2e^{-x}(-1)\\ &= xe^{-x}(2 - x)\end{aligned}$$

Since $e^{-x} > 0$ for all values of x, $\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$, and we consider intervals $x < 0$, $0 < x < 2$, and $x > 2$.

Value of x	Value of $\frac{dy}{dx} = xe^{-x}(2 - x)$	Graph Increasing or Decreasing
$x < 0$	$\frac{dy}{dx} < 0$	decreasing
$0 < x < 2$	$\frac{dy}{dx} > 0$	increasing
$x > 2$	$\frac{dy}{dx} < 0$	decreasing

Then $y = x^2e^{-x}$ is increasing on the interval $0 < x < 2$ and is decreasing on the intervals $x < 0$ and $x > 2$.

Exercise 9.1

Part A

Knowledge/ Understanding

1. Determine the points at which $f'(x) = 0$ for each of the following functions:

a. $f(x) = x^3 + 6x^2 + 1$ b. $f(x) = \sqrt{x^2 + 4}$

c. $f(x) = (2x - 1)^2(x^2 - 9)$ d. $f(x) = x^{\frac{2}{3}}(2x - 5)$

Communication

2. Explain how you would determine when a function is increasing or decreasing.

3. For each of the following functions, determine the direction of the curve by evaluating the derivative for a suitably large positive value of x.

a. $y = x^7 - 430x^6 - 150x^3$ b. $s = \frac{3t}{1 - t^2}$

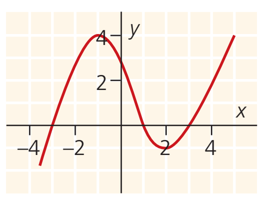
c. $y = x\ln x - x^4$ d. $y = 10xe^{-x} + x^2\ln x$

4. For each of the following graphs, state:

i) the intervals where the function is increasing.

ii) the intervals where the function is decreasing.

iii) the points where the tangent to the function is horizontal.

a.


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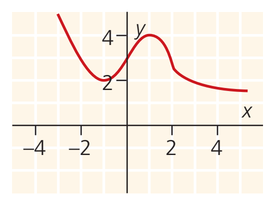
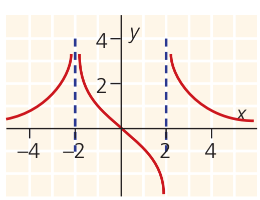
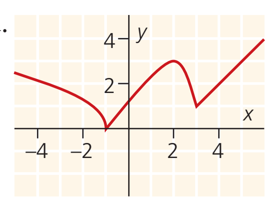


c.


d.


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5. Use a calculator to graph each of the following functions. Inspect the graph to estimate where the function is increasing and where it is decreasing. Verify your estimates with an analytic solution.

a. $f(x) = x^3 + 3x^2 + 1$ b. $f(x) = x^5 - 5x^4 + 100$

c. $f(x) = x + \frac{1}{x}$ d. $f(x) = \frac{x - 1}{x^2 + 3}$

e. $f(x) = x\ln(x)$ f. $f(x) = xe^{-x}$

Part B

6. Suppose that f is a differentiable function with derivative $f'(x) = (x - 1)(x + 2)(x + 3)$. Determine where the function values of f are increasing and where they are decreasing.

Application

7. Suppose that g is a differentiable function with derivative $g'(x) = (3x - 2)\ln(2x^2 - 3x + 2)$. Determine where the function values of f are increasing and where they are decreasing.

Application

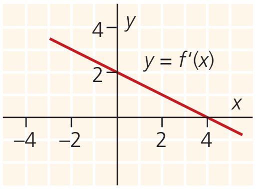
8. Sketch a graph of a function that is differentiable on the interval $-2 \leq x \leq 3$ and that satisfies the following conditions:
   - The graph of f passes through points $(-1, 0)$ and $(2, 5)$.
   - The function f is decreasing on $-2 < x < -1$, increasing on $-1 < x < 2$, and decreasing again on $2 < x < 5$.

9. Find constants a, b, and c that guarantee that the graph of $f(x) = x^3 + ax^2 + bx + c$ will increase to the point $(-3, 18)$, then decrease to the point $(1, -14)$, then continue increasing.

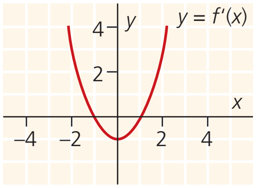
10. Sketch a graph of a function f that is differentiable and that satisfies the following conditions:
    - $f'(x) > 0$, when $x < -5$.
    - $f'(x) < 0$, when $-5 < x < 1$ and when $x > 1$.
    - $f'(-5) = 0$ and $f'(1) = 0$.
    - $f(-5) = 6$ and $f(1) = 2$.

11. Each of the following graphs represents the derivative function $f'(x)$ of a function $f(x)$. Determine
    i) the intervals where $f(x)$ is increasing.
    ii) the intervals where $f(x)$ is decreasing.
    iii) the x-coordinate for all local extrema of $f(x)$.
    iv) Assuming that $f(0) = 2$, make a rough sketch of the graph of each function.

a.

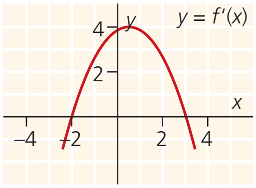


b.



$f'(x)$ is a quadratic function

c.



$f'(x)$ is a quadratic function

12. Use calculus techniques to show that the graph of the quadratic function $f(x) = ax^2 + bx + c$, $a > 0$, is decreasing on the interval $x < -\frac{b}{2a}$ and increasing on the interval $x > -\frac{b}{2a}$.

Part C

Thinking/Inquiry/ Problem Solving

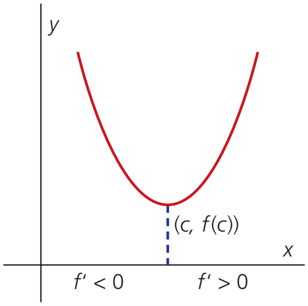
13. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both strictly increasing on $a \leq x \leq b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \leq x \leq b$, show that the product fg is also strictly increasing on $a \leq x \leq b$.

14. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both strictly increasing on $a \leq x \leq b$, and if $f(x) < 0$ and $g(x) < 0$ on $a \leq x \leq b$, is the product fg strictly increasing on $a \leq x \leq b$, strictly decreasing, or neither?

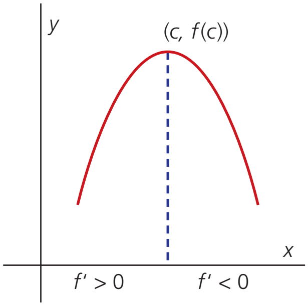
Section 9.2 — Critical Points, Relative Maxima, and Relative Minima

We saw in an earlier chapter that a maximum or minimum function value might be determined at a point $(c, f(c))$ if $f'(c) = 0$. Combining this with the properties of increasing and decreasing functions, we have a First Derivative Test for local extrema.

Test for local minimum and local maximum points. Suppose $f'(c) = 0$.



If $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point.



If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

There are possible implications of $f'(c) = 0$ other than the determination of maxima or minima. There are also simple functions for which the derivative does not exist at certain points. In Chapter 4, we demonstrated three different ways that this could happen.

EXAMPLE 1 For the function $y = x^4 - 8x^3 + 18x^2$, determine all values of x such that $f'(x) = 0$. For each of these values of x, determine whether it gives a relative maximum, a relative minimum, or neither for the function.

Solution

First determine $\frac{dy}{dx}$.

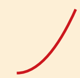
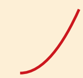
$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 24x^2 + 36x \\ &= 4x(x^2 - 6x + 9) \\ &= 4x(x - 3)^2\end{aligned}$$

For a relative maximum or minimum, let $\frac{dy}{dx} = 0$.

$4x(x - 3)^2 = 0$

$x = 0 \quad \text{or} \quad x = 3$

Both values of x are in the domain of the function. There is a horizontal tangent at each of these values of x. To determine which of the values of x yield relative maximum or minimum values of the function, we use a table to analyze the behaviour of $\frac{dy}{dx}$ and $y = x^4 - 8x^3 + 18x^2$.

	$x < 0$	$0 < x < 3$	$x > 3$
$\frac{dy}{dx}$	<0	>0	>0
$y = x^4 - 8x^3 + 18x^2$	decreasing	increasing	increasing
Shape of the Curve			

Using the information from the table, we see that at $x = 0$, there is a relative minimum value of the function, since the function values are decreasing before $x = 0$ and increasing after $x = 0$. We can also tell that at $x = 3$ there is neither a relative maximum nor minimum value, since the function values increase towards this point and increase away from it.

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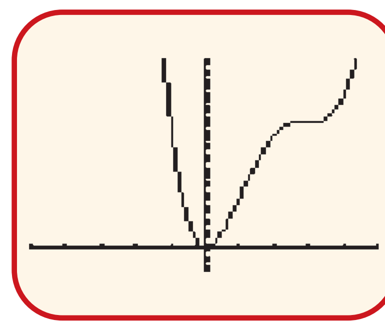
A calculator gives this graph for $y = x^4 - 8x^3 + 18x^2$, which verifies our analysis.



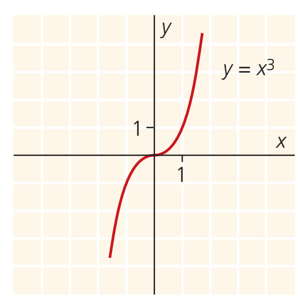
EXAMPLE 2

Determine whether or not the function $f(x) = x^3$ has a maximum or minimum at $(c, f(c))$ where $f'(c) = 0$.

Solution

The derivative is $f'(x) = 3x^2$.
Setting $f'(x) = 0$ gives

$$3x^2 = 0$$
$$x = 0.$$



From the graph, it is clear that (0, 0) is neither a maximum nor a minimum value of the function. Note that $f'(x) > 0$ for all values of x other than 0.

From this example, we can see that it is possible $f'(c) = 0$, but there is no maximum or minimum point at $(c, f(c))$. It is also possible for a maximum or minimum to occur at a point at which the derivative does not exist.

EXAMPLE 3 For the function $f(x) = (x + 2)^{\frac{2}{3}}$, determine values of x such that $f'(x) = 0$. Use your calculator to sketch a graph of the function.

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Solution

First determine $f'(x)$.

$$\begin{aligned} f'(x) &= \frac{2}{3}(x + 2)^{-\frac{1}{3}} \\ &= \frac{2}{3(x + 2)^{\frac{1}{3}}} \end{aligned}$$

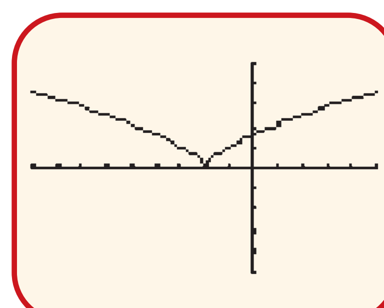
Note that there is no value of x for which $f'(x) = 0$, and $f'(x)$ is undefined for $x = -2$.

However, $x = -2$ is in the domain of $y = (x + 2)^{\frac{2}{3}}$, since $y = (-2 + 2)^{\frac{2}{3}} = 0$. Therefore, this function has one critical point, when $x = -2$. The slope of the tangent is undefined at this point. We determine the slopes of tangents for x values close to -2.

x	$f'(x) = \frac{2}{3(x+2)^{\frac{1}{3}}}$	x	$f'(x) = \frac{2}{3(x+2)^{\frac{2}{3}}}$
−2.1	−1.43629	−1.9	1.43629
−2.01	−3.09439	−1.99	3.09439
−2.001	−6.6667	−1.999	6.66667
−2.00001	−66.6667	−1.99999	66.6667

In this example, the slopes of the tangents to the left of $x = -2$ are approaching $-\infty$, while the slopes to the right of $x = -2$ are approaching $+\infty$. Since the slopes on opposite sides of $x = -2$ are not tending towards the same value, there is no tangent at $x = -2$ even though there is a point on the graph.

A calculator gives the following graph of $y = (x + 2)^{\frac{2}{3}}$. There is a cusp at $(-2, 0)$.



If a value c is *in the domain* of a function $f(x)$, and if this value is such that $f'(c) = 0$ or $f'(c)$ is undefined, then $(c, f(c))$ is called a *critical point* of the function f.

Notice that in the case of $f'(x) = 0$ at a critical point, the slope of the tangent is zero at that point and the tangent to the graph of $y = f(x)$ is horizontal.

In summary, critical points that occur when $\frac{dy}{dx} = 0$ give the locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists). Besides giving the location of interesting tangents, critical points also determine other interesting features for the graph of a function.

The value c determines the location of a relative (or local) minimum value for a function f if $f(c) < f(x)$ for all x near c.

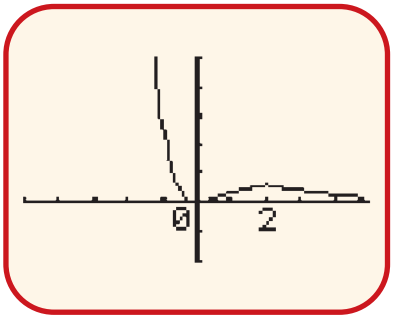
Similarly, the value c determines the location of a relative (or local) maximum value for a function f if $f(c) > f(x)$ for all x near c.

Together, relative maximum and minimum values of a function are called relative (or local) extrema.

Note that a relative minimum value of a function does not have to be the smallest value on the entire domain, just the smallest value in its neighbourhood. Similarly, a relative maximum value of a function does not have to be the largest value on the entire domain, just the largest value in its neighbourhood. Relative extrema occur graphically as peaks or valleys. The peaks can be either smooth or sharp.

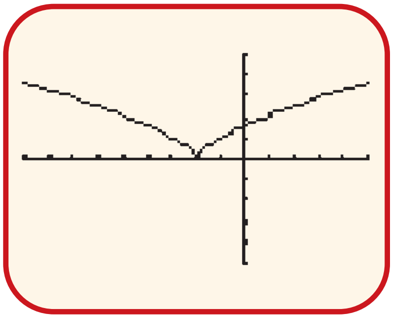
Let's now reconsider the graphs of two of the functions that we have already analyzed in applying this reasoning.

Here is the graph of $y = x^2e^{-x}$:



The function $f(x) = x^2e^{-x}$ has a relative maximum value at $x = 2$, since $f(2) = \frac{4}{e^2}$ is the largest value in its neighbourhood. However, $f(2) = \frac{4}{e^2}$ is not the largest function value on the entire domain. The function $f(x) = x^2e^{-x}$ also has a relative minimum value at $x = 0$, since $f(0) = 0$ is the smallest value in its neighbourhood. This value also happens to be the smallest on the entire domain.

Here is the graph of $y = (x + 2)^{\frac{2}{3}}$:



The function $f(x) = (x + 2)^{\frac{2}{3}}$ has a relative minimum value at $x = -2$, which also happens to be a critical value of the function.

Every relative maximum or minimum value of a function occurs at a critical point of the function.

In simple terms, peaks or valleys occur on the graph of a function at places where the tangent to the graph is horizontal, vertical, or does not exist.

How do we determine whether a critical point yields a relative maximum or minimum value of a function without examining the graph of the function? We use the first derivative to analyze whether the function is increasing or decreasing on either side of the critical point.

Algorithm for finding relative maximum and minimum values of a function *f*:

1. **Find critical points of the function; that is, determine where $f'(x) = 0$ and where $f'(x)$ is undefined, for x values in the domain of f.**
2. **Use the first derivative to analyze whether f is increasing or decreasing on either side of each critical point.**
3. **Conclude that each critical point locates either a relative maximum value of the function f, a relative minimum value, or neither.**

Exercise 9.2

Part A

Communication

1. Explain what it means to determine the critical points of the graph of a given function.

Knowledge/Understanding

2. a. For the function $y = x^3 - 6x^2$, explain how you would find the critical points.

b. Determine the critical points for $y = x^3 - 6x^2$ and then sketch the graph.

3. For each of the following, find the critical points. Use the first derivative test to determine whether the critical point is a local maximum, local minimum, or neither.

a. $y = x^4 - 8x^2$ b. $f(x) = \frac{2x}{x^2 + 9}$

c. $y = xe^{-4x}$ d. $y = \ln(x^2 - 3x + 4)$

4. For each of the parts in Question 3, find the x- and y-intercepts and then sketch the curve.

5. Find the critical points for each of the following. Determine whether the critical point is a local maximum or minimum and whether or not the tangent is parallel to the horizontal axis.

a. $h(x) = -6x^3 + 18x^2 + 3$ b. $s = -t^2e^{-3t}$

c. $y = (x - 5)^{\frac{1}{3}}$ d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

e. $g(t) = t^5 + t^3$ f. $y = x^2 - 12x^{\frac{1}{3}}$

technology

6. Use a technology of your choice to graph the functions in Question 5.

Part B

Knowledge/Understanding

7. Find the critical points for each of the following functions, and determine whether the function has a relative maximum value, a relative minimum value, or neither at the critical points. Sketch the graph of each function.

a. $f(x) = -2x^2 + 8x + 13$ b. $f(x) = \frac{1}{3}x^3 - 9x + 2$

c. $f(x) = 2x^3 + 9x^2 + 12x$ d. $f(x) = -3x^3 - 5x$

e. $f(x) = \sqrt{x^2 - 2x + 2}$ f. $f(x) = 3x^4 - 4x^3$

g. $f(x) = e^{-x^2}$ h. $f(x) = x^2\ln x$

8. Suppose f is a differentiable function with derivative $f'(x) = (x + 1)(x - 2)(x + 6)$. Find all critical numbers of f and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

Application

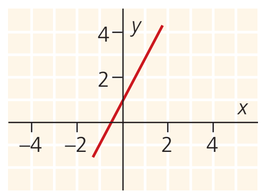
9. Sketch a graph of a function f that is differentiable on the interval $-3 \le x \le 4$ and that satisfies the following conditions:

- The function f is decreasing on $-1 < x < 3$ and increasing elsewhere on $-3 \le x \le 4$.

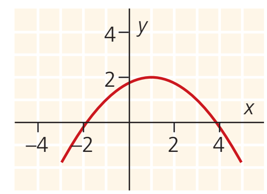
- The largest value of f is 6 and the smallest is 0.
- The graph of f has relative extrema at $(-1, 6)$ and $(3, 1)$.

10. Find values for a, b, and c such that the graph of $y = ax^2 + bx + c$ has a relative maximum at $(3, 12)$ and crosses the y-axis at $(0, 1)$.

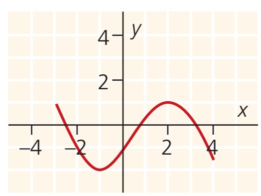
11. For each of the following graphs of the function $y = f(x)$, make a rough sketch of the derivative function $f'(x)$. By comparing the graphs of $f(x)$ and $f'(x)$, show that the intervals for which $f(x)$ is increasing correspond to the intervals where $f'(x)$ is positive. Also show that the intervals where $f(x)$ is decreasing correspond to the intervals for which $f'(x)$ is negative.

a.


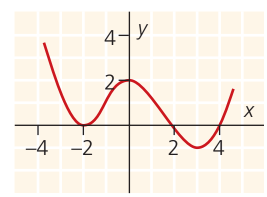
$f(x)$ is a linear function.

b.


$f(x)$ is a quadratic function.

c.


$f(x)$ is a cubic function.

d.


$f(x)$ is a quartic function.

12. For the function $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$,

   a. find constants a, b, c, and d that guarantee that the graph of f will have horizontal tangents at $(-2, -73)$ and $(0, -9)$.

   b. there is a third point that has a horizontal tangent. Find this point.

   c. For all three points, determine whether each corresponds to a relative maximum, a relative minimum, or neither.

Part C

13. For each of the following polynomials, find the local extrema and the direction the curve is opening for $x = 100$. Use this information to make a quick sketch of the curve.

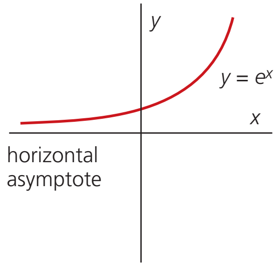
   a. $y = 4 - 3x^2 - x^4$

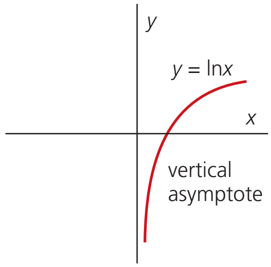
   b. $y = 3x^5 - 5x^3 - 30x$

14. Suppose that $f(x)$ and $g(x)$ are positive functions such that $f(x)$ has a local maximum and $g(x)$ has a local minimum at $x = c$. Show that $h(x) = \frac{f(x)}{g(x)}$ has a local maximum at $x = c$.

Section 9.3 — Vertical and Horizontal Asymptotes

In sketching the graph of the exponential function $y = e^x$, you saw that the x-axis is a horizontal asymptote. In graphing $y = \ln x$, you saw that the y-axis is a vertical asymptote.





Asymptotes play a significant role in curve sketching. In this section, we will consider vertical and horizontal asymptotes of rational functions and expand our understanding of asymptotes of exponential and logarithmic functions.

Vertical Asymptotes and Rational Functions

INVESTIGATION The purpose of this investigation is to examine the occurrence of vertical asymptotes for rational functions.

1. Use your graphing calculator to obtain the graph of $f(x) = \frac{1}{x - k}$ and the table of values for each of the following: $k = 3, 1, 0, -2, -4,$ and -5.
2. Describe the behaviour of each graph as x approaches k from the right and from the left.
3. Repeat Questions 1 and 2 for the function $f(x) = \frac{x + 3}{x - k}$, using the same values of k.
4. Repeat Questions 1 and 2 for the function $f(x) = \frac{1}{x^2 + x - k}$, using the following values: $k = 2, 6,$ and 12.
5. Make a general statement about the existence of a vertical asymptote for a rational function of the form $y = \frac{p(x)}{q(x)}$ if there is a value c such that $q(c) = 0$ and $p(c) \neq 0$.

We use the notation $x \to c^+$ to indicate that x approaches c from the right. Similarly, $x \to c^-$ means that x approaches c from the left.

You can see from this investigation that as $x \to c$ from either side, the function values get increasingly large and positive or negative depending on the value of $p(c)$. We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers. They are symbols that represent the value of a function that increases or decreases without limits.

Because the symbol ∞ is not a number, the limits

$$\lim_{x \to c^+} \frac{1}{x-c} \quad \text{and} \quad \lim_{x \to c^-} \frac{1}{x-c}$$

do not exist. However, for convenience, we use the notation

$$\lim_{x \to c^+} \frac{1}{x-c} = +\infty \quad \text{and} \quad \lim_{x \to c^-} \frac{1}{x-c} = -\infty.$$

In similar fashion, and for the same reason, we also write

$$\lim_{x \to 0^+} \ln x = -\infty.$$

These three limits form the basis for determining asymptotes to simple functions.

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

EXAMPLE 1 Determine any vertical asymptotes of the function $f(x) = \frac{x}{x^2 + x - 2}$, and describe the behaviour of the graph of the function for values of x near the asymptotes.

Solution

First determine the values of x for which $f(x)$ is undefined by solving

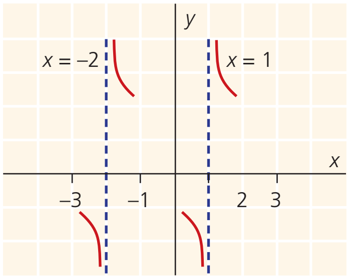
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2 \text{ or } x = 1.$$

Neither of these values for x makes the numerator zero, so both of these values for x give vertical asymptotes. The asymptotes are $x = -2$ and $x = 1$.

To determine the behaviour of the graph near the asymptotes, it can be helpful to use a chart.

x-values	x	$x + 2$	$x - 1$	$f(x)$	$\lim_{x \to c} f(x)$
$x \to -2^-$	<0	<0	<0	<0	$-\infty$
$x \to -2^+$	<0	>0	<0	>0	$+\infty$
$x \to 1^-$	>0	>0	<0	<0	$-\infty$
$x \to 1^+$	>0	>0	>0	>0	$+\infty$

The behaviour of the graph can be illustrated as follows:



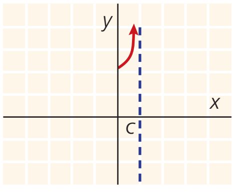
To proceed beyond this point, we require additional information.

Vertical Asymptotes

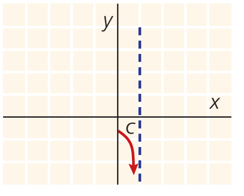
The graph of $f(x)$ has a vertical asymptote $x = c$ if one of the following limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty \quad \lim_{x \to c^-} f(x) = -\infty \quad \lim_{x \to c^+} f(x) = +\infty \quad \lim_{x \to c^+} f(x) = -\infty$$

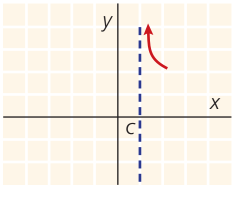
The following graphs correspond to each limit statement.



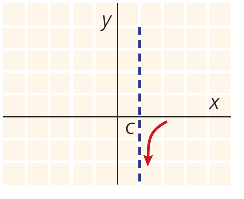
$\lim_{x \to c^-} f(x) = +\infty$



$\lim_{x \to c^-} f(x) = -\infty$



$\lim_{x \to c^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

Horizontal Asymptotes

Consider the behaviour of rational functions $f(x) = \frac{p(x)}{q(x)}$ as x increases without bound, in both the positive and negative directions. The following notation is used to describe this behaviour:

$$\lim_{x \to +\infty} f(x) \quad \text{and} \quad \lim_{x \to -\infty} f(x).$$

The notation $x \to +\infty$ is read as "x tends to positive infinity" and means that the values of x are positive and growing without bound. Similarly, the notation

$x \to -\infty$ is read as "x tends to negative infinity" and means that the values of x are negative and growing in magnitude without bound.

The value of these limits can be determined by making two observations. The first is a list of simple limits parallel to those used in determining vertical asymptotes.

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

As observed earlier, for the exponential function $f(x) = e^x$,

$$\lim_{x\to-\infty} e^x = 0.$$

The second observation is that a polynomial can always be written so that the term of highest degree is a factor.

EXAMPLE 2

Write each of the following so that the term of highest degree is a factor.

a. $p(x) = x^2 + 4x + 1$

b. $q(x) = 3x^2 - 4x + 5$

Solution

a. $p(x) = x^2 + 4x + 1$
$= x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)$

b. $q(x) = 3x^2 - 4x + 5$
$= 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right)$

The value of writing a polynomial in this form is clear. It is easy to see that as x becomes large, either positive or negative, the value of the second factor always approaches 1.

We can now determine the limit of a rational function in which the degree of $p(x)$ is equal to or less than the degree of $q(x)$.

EXAMPLE 3

Determine the value of each of the following:

a. $\lim_{x\to+\infty} \frac{2x-3}{x+1}$

b. $\lim_{x\to-\infty} \frac{x}{x^2+1}$

c. $\lim_{x\to+\infty} \frac{2x^2+3}{3x^2-x+4}$

Solution

a.
$$f(x) = \frac{2x-3}{x+1} = \frac{2x\left(1-\frac{3}{2x}\right)}{x\left(1+\frac{1}{x}\right)}$$
$$= \frac{2\left(1-\frac{3}{2x}\right)}{1+\frac{1}{x}}$$
$$\lim_{x\to+\infty} f(x) = \frac{2\left[\lim_{x\to+\infty}\left(1-\frac{3}{2x}\right)\right]}{\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)}$$
$$= \frac{2(1-0)}{1+0}$$
$$= 2$$

b.
$$g(x) = \frac{x}{x^2+1}$$
$$= \frac{x(1)}{x^2\left(1+\frac{1}{x^2}\right)}$$
$$\lim_{x\to-\infty} g(x) = \frac{1}{\lim_{x\to-\infty} x \bullet \lim_{x\to-\infty}\left(1+\frac{1}{x^2}\right)}$$
$$= \lim_{x\to-\infty} \frac{1}{x}$$
$$= 0$$

c. $p(x) = \dfrac{2x^2 + 3}{3x^2 - x + 4}$

$$= \frac{2x^2\left(1 + \frac{3}{2x^2}\right)}{}$$

$$= \frac{2\left(1 + \frac{3}{2x^2}\right)}{}$$

$$\lim_{x \to +\infty} p(x) = \frac{2}{3}\,\frac{\lim\limits_{x \to +\infty}\left(1 + \frac{3}{2x^2}\right)}{\lim\limits_{x \to +\infty}\left(1 - \frac{1}{3x} + \frac{4}{3x^2}\right)}$$

$$= \frac{2}{3}\,\frac{(1 + 0)}{(1 - 0 + 0)}$$

$$= \frac{2}{3}$$

Alternate Solution

Divide the numerator and denominator by the largest power of x, that is, x^2.

$$p(x) = \frac{2 + \frac{3}{x^2}}{3 - \frac{1}{x} + \frac{4}{x^2}}$$

$$\lim_{x \to +\infty} p(x) = \frac{\lim\limits_{x \to +\infty}\left(2 + \frac{3}{x^2}\right)}{\lim\limits_{x \to +\infty}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}$$

$$= \frac{2}{3}$$

When $\lim\limits_{x \to +\infty} f(x) = k$ or $\lim\limits_{x \to -\infty} f(x) = k$, the graph of the function is approaching the line $y = k$. This line is a horizontal asymptote of the function. In Example 3, $y = 2$ is a horizontal asymptote of $f(x) = \dfrac{2x - 3}{x + 1}$.

From the solution above, you can see that

$$\lim_{x \to -\infty} \frac{2x - 3}{x + 1} = \frac{2\left(\lim\limits_{x \to -\infty}\left(1 - \frac{3}{x}\right)\right)}{\lim\limits_{x \to -\infty}\left(1 + \frac{1}{x}\right)}$$

$$= 2$$

and, therefore, $y = 2$ is an asymptote for large positive x-values and also for large negative x-values.

In order to use this knowledge in sketching the graph for this function, we need to know whether the curve approaches the asymptote from above or from below. This is answered by considering $f(x) - k$ where k is the limit determined. This is illustrated in the following examples.

EXAMPLE 4 Determine the equations of any horizontal asymptotes of the function $f(x) = \dfrac{3x + 5}{2x - 1}$, and state whether the graph approaches the asymptote from above or below.

Solution

$$f(x) = \frac{3x + 5}{2x - 1} = \frac{3x\left(1 + \frac{5}{3x}\right)}{}$$

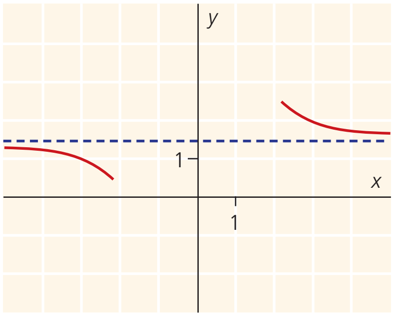
$$= \frac{3\left(1 + \frac{5}{3x}\right)}{}$$

$$\lim_{x \to +\infty} f(x) = \frac{3 \lim_{x \to +\infty}\left(1 + \frac{5}{3x}\right)}{2 \lim_{x \to +\infty}\left(1 - \frac{1}{2x}\right)}$$
$$= \frac{3}{2}$$

Similiarly, we can show $\lim_{x \to -\infty} f(x) = \frac{3}{2}$. Then $y = \frac{3}{2}$ is a horizontal asymptote of the graph $f(x)$ for both large positive and negative values of x. To determine whether the graph approaches the asymptote from above or below, we consider very large positive and negative values for x.

If x is large and positive, for example, if $x = 1000$, $f(x) = \frac{3005}{1999}$, which is greater than $\frac{3}{2}$. Therefore, the graph approaches the asymptote $y = \frac{3}{2}$ from above.

If x is large and negative, for example, if $x = -1000$, $f(x) = \frac{-2995}{-2000}$, which is less than $\frac{3}{2}$. This graph approaches the asymptote $y = \frac{3}{2}$ from below, as illustrated in the diagram.



EXAMPLE 5 For the function $f(x) = \frac{3x}{x^2 - x - 6}$, determine the equations of all horizontal or vertical asymptotes and illustrate the behaviour of the graph as it approaches the asymptotes.

Solution

For vertical asymptotes

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \quad \text{or} \quad x = -2$$

There are two vertical asymptotes at $x = 3$ and $x = -2$.

x-values	x	$x-3$	$x+2$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to 3^-$	>0	<0	>0	<0	$-\infty$
$x \to 3^+$	>0	>0	>0	>0	$+\infty$
$x \to -2^-$	<0	<0	<0	<0	$-\infty$
$x \to -2^+$	<0	<0	>0	>0	$+\infty$

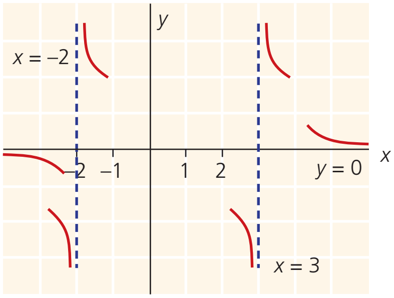
For horizontal asymptotes,

$$\begin{aligned} f(x) &= \frac{3x}{x^2 - x - 6} \\ &= \frac{3x}{x^2\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \\ &= \frac{3}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \end{aligned}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty} \frac{3}{x} = 0.$$

Similarly, we can show $\lim_{x\to-\infty} f(x) = 0$. Therefore, y = 0 is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x.
As x becomes large positively, $f(x) > 0$, so the graph is above the asymptote.
As x becomes large negatively, $f(x) < 0$, so the graph is below the asymptote.

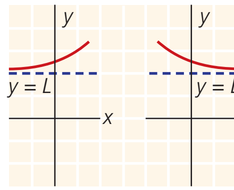
This diagram illustrates the behaviour of the graph as it nears the asymptotes:



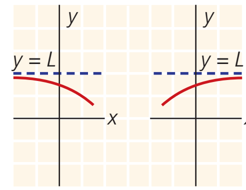
Horizontal Asymptotes

If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote of the graph of $f(x)$.

The following graphs illustrate some typical situations:



$f(x) > L$ for large x



$f(x) < L$ for large x

In addition to vertical and horizontal asymptotes, it is possible for a graph to have **oblique asymptotes**. These are straight lines that are not parallel to the axes and that the curve approaches infinitely closely. They occur with rational functions in which the degree of the numerator polynomial exceeds the degree of the denominator polynomial by exactly one. This is illustrated in the following example.

EXAMPLE 6 Determine the equations of all asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.

Solution

Since $x + 1 = 0$ for $x = -1$ and $2x^2 + 3x - 1 \neq 0$ for $x = -1$, then $x = -1$ is a vertical asymptote.

$$\text{Now } \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} = \lim_{x\to\infty} \frac{2x^2\left(1 + \frac{3}{2x} - \frac{1}{2x^2}\right)}{}$$

$$= \lim_{x\to\infty} 2x.$$

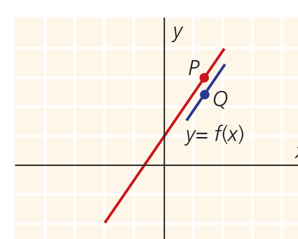
This limit does not exist, and by a similar calculation, $\lim_{x\to-\infty} f(x)$ does not exist, so there is no horizontal asymptote.

After dividing the numerator by the denominator,

$$\begin{array}{r} 2x + 1 \\ x + 1\overline{)2x^2 + 3x - 1} \\ \underline{2x^2 + 2x} \\ x - 1 \\ \underline{x + 1} \\ -2 \end{array}$$

we can write $f(x)$ in the form $f(x) = 2x + 1 - \frac{2}{x + 1}$.

Now consider the straight line $y = 2x + 1$ and the graph of $y = f(x)$. For any value of x, we determine point $P(x, 2x + 1)$ on the line and $Q\left(x, 2x + 1 - \frac{2}{x + 1}\right)$ on the curve.



Then the vertical distance QP from the curve to the line is $QP = 2x + 1 - \left(2x + 1 - \frac{2}{x + 1}\right)$

$$= \frac{2}{x + 1}.$$

$$\text{Then } \lim_{x \to \infty} QP = \lim_{x \to \infty} \frac{2}{x + 1}$$
$$= 0.$$

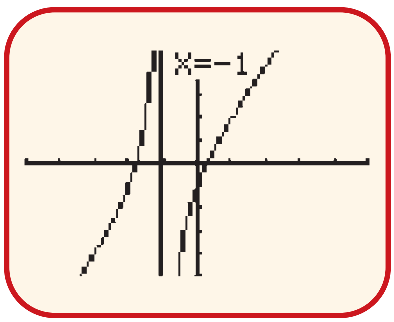
That is, as x gets very large, the curve approaches the line but never touches it. Therefore, the line $y = 2x + 1$ is an asymptote of the curve.

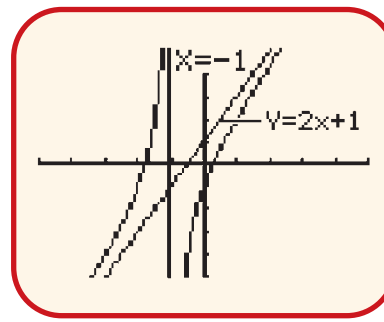
Since $\lim_{x \to -\infty} \frac{2}{x + 1} = 0$, the line is also an asymptote for large negative values of x.

In conclusion, there are two asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$. They are $y = 2x + 1$ and $x = -1$.

technology

Use the graphing calculator to obtain the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.





Note that the vertical asymptote $x = -1$ appears on the graph on the left, but the oblique asymptote $y = 2x + 1$ does not. Use the Y_2 function to graph the oblique asymptote $y = 2x + 1$.

The techniques for curve sketching developed to this point are described in the following algorithm. As we develop new ideas, the algorithm will be extended.

Algorithm for Curve Sketching

To sketch a curve, apply the following in the order shown. Add the information to the diagram step by step.

Step 1: Check for any discontinuities in the domain. Determine if there are vertical asymptotes and the direction at which the curve approaches these asymptotes.
Step 2: Find the y-intercept.
Step 3: Find any critical points.
Step 4: Use the first derivative test to determine the type of critical points that may be present.
Step 5: Extremity tests: Determine $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$.

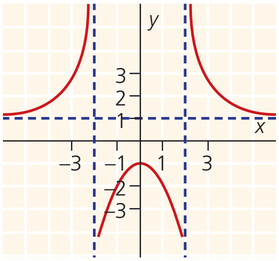
Exercise 9.3

Part A

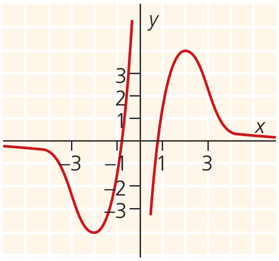
Knowledge/ Understanding

1. State the equations of the vertical and horizontal asymptotes of the curves shown.

a.



b.



Communication

2. Under what condition does a rational function have vertical, horizontal, and oblique asymptotes?

3. Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$, using the symbol "∞" when appropriate.

a. $f(x) = \frac{2x + 3}{x - 1}$

b. $f(x) = \frac{5x^2 - 3}{x^2 + 2}$

c. $f(x) = \frac{-5x^2 + 3x}{2x^2 - 5}$

d. $f(x) = \frac{2x^5 - 3x^2 + 5}{3x^4 + 5x - 4}$

4. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \frac{x}{x + 5}$

b. $f(x) = \frac{x + 2}{x - 2}$

c. $s = \frac{1}{(t - 3)^2}$

d. $y = \frac{x^2 - x - 6}{x - 3}$

e. $g(x) = \frac{1}{e^x - 2}$

f. $y = x\ln x$

5. For each of the following, determine the equations of any horizontal asymptotes and state whether the curve approaches the asymptote from above or below.

a. $y = \frac{x}{x + 4}$

b. $f(x) = \frac{2x}{x^2 - 1}$

c. $g(t) = \frac{3t^2 + 4}{t^2 - 1}$

d. $y = \frac{3x^2 - 8x - 7}{x - 4}$

Part B

technology

6. For each of the following, check for discontinuities and then use at least two other tests to make a rough sketch of the curve. Verify on a calculator.

a. $y = \frac{x - 3}{x + 5}$

b. $f(x) = \frac{5}{(x + 2)^2}$

c. $g(t) = \frac{t^2 - 2t - 15}{t - 5}$

d. $p(x) = \frac{15}{6 - 2e^x}$ e. $y = \frac{(2 + x)(3 - 2x)}{(x^2 - 3x)}$ f. $P = \frac{10}{n^2 + 4}$

7. Find the equation of the oblique asymptote for each of the following:
   a. $f(x) = \frac{3x^2 - 2x - 17}{x - 3}$
   b. $f(x) = \frac{2x^2 + 9x + 2}{2x + 3}$
   c. $f(x) = \frac{x^3 - 1}{x^2 + 2x}$
   d. $f(x) = \frac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

8. a. In Question 7 **a,** determine whether the curve approaches the asymptote from above or below.
   b. In Question 7 **b,** determine the direction from which the curve approaches the asymptote.

Application

9. Use the algorithm for curve sketching to sketch the following:
   a. $f(x) = \frac{3 - x}{2x + 5}$
   b. $h(t) = 2t^3 - 15t^2 + 36t - 10$
   c. $y = \frac{20}{x^2 + 4}$
   d. $s(t) = t + \frac{1}{t}$
   e. $g(x) = \frac{2x^2 + 5x + 2}{x + 3}$
   f. $s(t) = \frac{t^2 + 4t - 21}{t - 3}, t \geq -7$

10. For the function $y = \frac{ax + b}{cx + d}$, where a, b, c, and d are constants, $a \neq 0$, $c \neq 0$,
   a. determine the horizontal asymptote of the graph.
   b. determine the vertical asymptote of the graph.

Part C

Thinking/Inquiry/Problem Solving

11. Find constants a and b that guarantee that the graph of the function defined by $f(x) = \frac{ax + 5}{3 - bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

12. This question will illustrate that we cannot work with the symbol "∞" as though it were a real number. Consider the functions $f(x) = \frac{x^2 + 1}{x + 1}$ and $g(x) = \frac{x^2 + 2x + 1}{x + 1}$.
   a. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$.
   b. Evaluate $\lim_{x \to +\infty} [f(x) - g(x)]$ and show that the limit is not zero.

13. Use the algorithm for curve sketching to sketch the function $f(x) = \frac{2x^2 - 2x}{x^2 - 9}$.

14. Determine the oblique asymptote of the graph of $y = \frac{x^2 + 3x + 7}{x + 2}$.

Section 9.4 — Concavity and Points of Inflection

In Chapter 5, you saw that the second derivative of a function has applications in problems involving velocity and acceleration, or in general rates of change problems. Here we examine the use of the second derivative of a function in curve sketching.

INVESTIGATION 1 The purpose of this investigation is to examine the relationship between tangent slopes and the second derivative of a function.

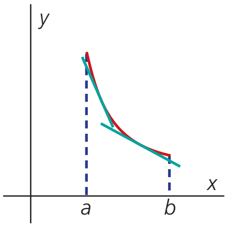
1. Sketch the graph of $f(x) = x^2$.
2. Determine the slope of the tangent to the curve at each of the points having $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$, and sketch each of these tangents.
3. Are the tangents above or below the graph of $y = f(x)$?
4. Describe the change in the slopes as x increases.
5. Determine $f''(x)$ and compare it with your answer in Question 3.
6. Repeat Questions 2, 3, and 4 for the graph of $f(x) = -x^2$.
7. How does the value of $f''(x)$ relate to the way in which the curve opens?

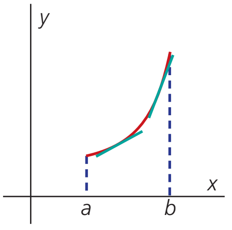
INVESTIGATION 2 The purpose of this investigation is to extend the results of Investigation 1 to other functions.

1. Sketch the graph of $f(x) = x^3$.
2. Determine all values of x for which $f'(x) = 0$.
3. Determine intervals of the domain of the function such that $f''(x) < 0$, $f''(x) = 0$, and $f''(x) > 0$.
4. For values of x such that $f''(x) < 0$, how does the shape of the curve compare to your conclusions in Investigation 1?
5. Repeat Question 4 for values of x such that $f''(x) > 0$.
6. What happens when $f''(x) = 0$?
7. Using your observations from this investigation, sketch the graph of $y = x^3 - 12x$.

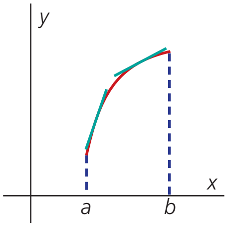
From these investigations, we can make a summary of the behaviour of the graphs.

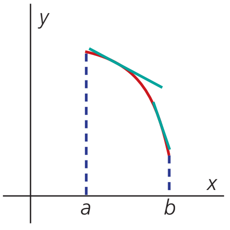
1. The graph of $y = f(x)$ is *concave up* on an interval $a \le x \le b$ in which its slopes are increasing. On this interval, $f''(x)$ exists and $f''(x) > 0$. The graph of the function is above the tangent at every point in the interval.





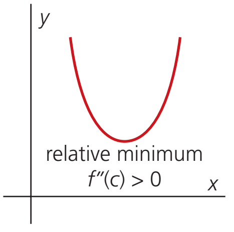
2. The graph of $y = f(x)$ is *concave down* on an interval $a \le x \le b$ in which its slopes are decreasing. On this interval, $f''(x)$ exists and $f''(x) < 0$. The graph of the function is below the tangent at every point in the interval.



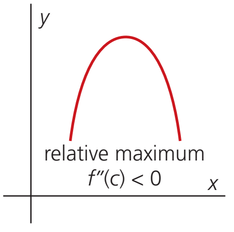


3. If $y = f(x)$ has a critical point at $x = c$, with $f'(c) = 0$, then:

   a. the graph is concave up and $x = c$ is the location of a relative minimum value of the function if $f''(c) > 0$.

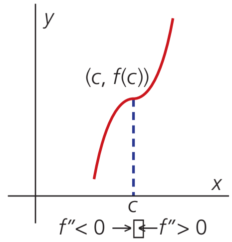


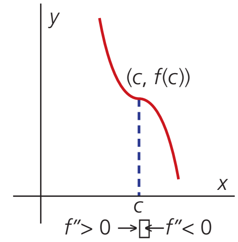
   b. the graph is concave down and $x = c$ is the location of a relative maximum value of the function if $f''(c) < 0$.



   c. the nature of the critical point cannot be determined if $f''(c) = 0$ without further work.

4. A point of inflection occurs at $(c, f(c))$ on the graph of $y = f(x)$ if $f''(x)$ changes sign at $x = c$. That is, the curve changes from concave down to concave up, or vice versa.





5. All points of inflection of the graph of $y = f(x)$ must occur either for $\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$, undefined.

In the following examples, we can use these properties to sketch graphs of other functions.

EXAMPLE 1

Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$.

Solution

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Setting $\frac{dy}{dx} = 0$, we obtain
$$3(x^2 - 2x - 3) = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \quad \text{or} \quad x = -1.$$

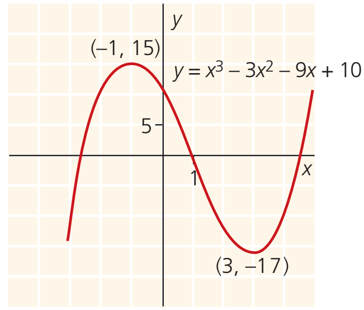
$$\frac{d^2y}{dx^2} = 6x - 6$$

Setting $\frac{d^2y}{dx^2} = 0$, we obtain
$$6x - 6 = 0$$
$$x = 1.$$

Now determine the sign of $f''(x)$ in the intervals determined by $x = 1$.

Interval	$x < 1$	$x = 1$	$x > 1$
$f''(x)$	< 0	0	> 0
Graph of $f(x)$	concave down	point of inflection	concave up
Sketch of $f(x)$			

Using $x = 3$, we obtain the local minimum point, $(3, -17)$.
Using $x = -1$, we obtain the local maximum point, $(-1, 15)$.
The graph can now be sketched.



EXAMPLE 2 Sketch the graph of $f(x) = x^4$.

technology

Solution

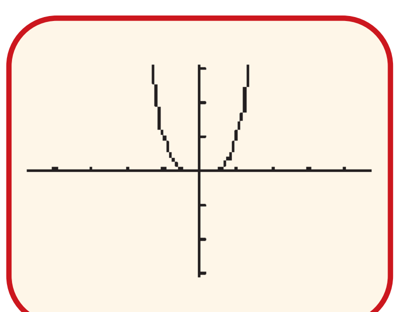
The first and second derivatives of $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$.
Setting $f''(x) = 0$, we obtain $12x^2 = 0$

$$x = 0.$$

But $x = 0$ is also obtained from $f'(x) = 0$.
Now determine the sign of $f''(x)$ in the intervals determined by $x = 0$.

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	> 0
Graph of $f(x)$	concave up	?	concave up
Sketch of $f(x)$			

We conclude that the point (0, 0) is *not* an inflection point because $f''(x)$ does not change sign at $x = 0$.



EXAMPLE 3

Sketch the graph of the function $f(x) = x^{\frac{1}{3}}$.

Solution

The derivative of $f(x)$ is

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$
$$= \frac{1}{3x^{\frac{2}{3}}}.$$

Note that $f'(0)$ does not exist, so that $x = 0$ is a critical value of $f(x)$. It is important to determine the behaviour of $f'(x)$ as $x \to 0$. Since $f'(x) > 0$ for all $x \neq 0$ and the denominator of $f'(x)$ is zero when $x = 0$, we have

$$\lim_{x\to 0} f'(x) = +\infty.$$

This means that there is a *vertical tangent* at $x = 0$. In addition, $f(x)$ is increasing for $x < 0$ and $x > 0$.

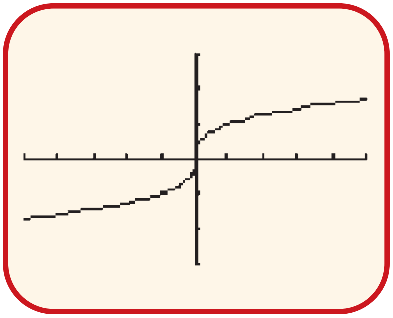
The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}.$$

Since $x^{\frac{5}{3}} > 0$ if $x > 0$ and $x^{\frac{5}{3}} < 0$ for $x < 0$, we obtain the following table:

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	+	does not exist	–
$f(x)$			

The graph has a point of inflection when $x = 0$, even though $f'(0)$ and $f''(0)$ do not exist. Note that the curve crosses its tangent at $x = 0$.



EXAMPLE 4

Determine any points of inflection in the graph of $f(x) = \frac{1}{x^2 + 3}$.

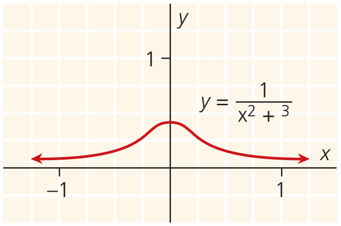
Solution

The derivative of $f(x) = \frac{1}{x^2 + 3} = (x^2 + 3)^{-1}$ is

$$f'(x) = -2x(x^2 + 3)^{-2}.$$

The second derivative is

$$\begin{aligned} f''(x) &= -2(x^2+3)^{-2} + 4x(x^2+3)^{-3}(2x) \\ &= \frac{-2}{(x^2+3)^2} + \frac{8x^2}{(x^2+3)^3} \\ &= \frac{-2(x^2+3)+8x^2}{(x^2+3)^3} \\ &= \frac{6x^2-6}{(x^2+3)^3}. \end{aligned}$$



Setting $f''(x) = 0$ gives $6x^2 - 6 = 0$

$$x = \pm 1.$$

Determine the sign of $f''(x)$ in the intervals determined by $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	concave up	point of inflection	concave down	point of inflection	concave up

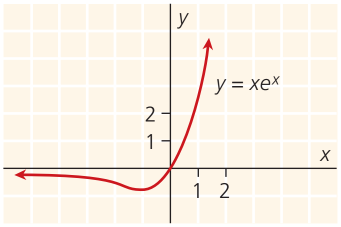
Therefore, $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$ are points of inflection in the graph of $f(x)$.

EXAMPLE 5 Determine any points of inflection in the graph of $f(x) = xe^x$.

Solution

The derivative of $f(x) = xe^x$ is

$$f'(x) = e^x + xe^x.$$



The second derivative is

$$\begin{aligned} f''(x) &= e^x + e^x + xe^x \\ &= e^x(2 + x). \end{aligned}$$

Setting $f''(x) = 0$ gives $e^x(2 + x) = 0$

$$x + 2 = 0 \quad (\text{since } e^x \neq 0 \text{ for any } x)$$

$$x = -2.$$

Determine the sign of $f''(x)$ in intervals determined by $x = -2$.

Interval	$x < -2$	$x = -2$	$x > -2$
$f''(x)$	< 0	$= 0$	> 0
Graph of $f(x)$	concave down	point of inflection	concave up

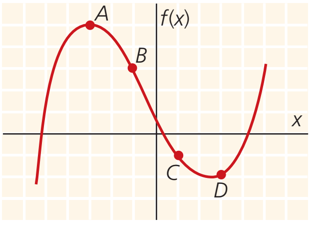
Therefore, $\left(-2, -\frac{2}{e^2}\right)$ is a point of inflection in the graph of $f(x)$.

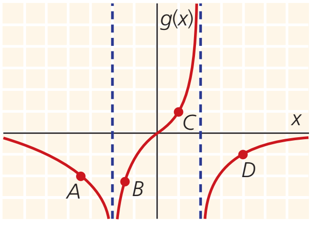
Exercise 9.4

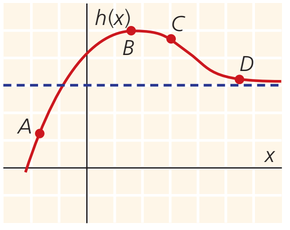
Part A

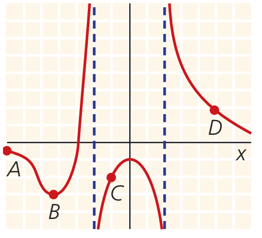
Knowledge/ Understanding

1. For each of the following functions, state whether the value of the second derivative is positive or negative at each of points A, B, C, and D.

a.


b.


c.


d.


2. Find the critical points for each of the following, and use a second derivative test to decide if the point is a local maximum, a local minimum, or neither.

a. $y = x^3 - 6x^2 - 15x + 10$

b. $y = \dfrac{25}{x^2 + 48}$

c. $s = t + t^{-1}$

d. $y = (x - 3)^3 + 8$

3. For Question 2, determine the points of inflection for each of the given functions. In each case, conduct a test to determine the change of sign in the second derivative.

4. Find the value of the second derivative at the value indicated. Determine whether the curve lies above or below the tangent at this point.

a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

b. $g(x) = x^2 - \dfrac{1}{x}$ at $x = -1$

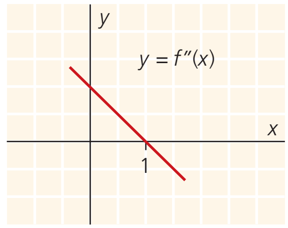
c. $s = e^t \ln t$ at $t = 1$

d. $p = \dfrac{w}{\quad\quad}$ at $w = 3$

Part B

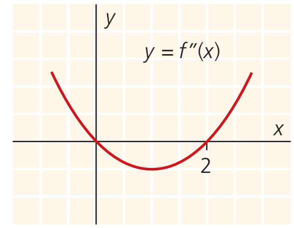
5. Each of the following graphs represents the second derivative, $f''(x)$, of a function $f(x)$.

i)



$f''(x)$ is a linear function

ii)



$f''(x)$ is a quadratic function

a. On which intervals is the graph of $f(x)$ concave up, and on which is the graph concave down?

b. List the x-coordinates of all points of inflection.

c. Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 2$.

Communication

6. Describe how you would use the second derivative to determine the local minimum or maximum.

7. In the algorithm for curve sketching on page 360, reword Step 4 to include the use of the second derivative to test for local minimum or maximum values.

8. For each of the following functions,

   i) determine any points of inflection, and

   ii) use the results of part **i** along with the revised algorithm to sketch each function.

   a. $f(x) = x^4 + 4x^3$

   b. $y = x - \ln x$

   c. $y = e^x + e^{-x}$

   d. $g(w) = \dfrac{4w^2 - 3}{w^3}$

9. Sketch the graph of a function with the following properties:

- $f'(x) > 0$ when $x < 2$ and when $2 < x < 5$
- $f'(x) < 0$ when $x > 5$
- $f'(2) = 0$ and $f'(5) = 0$
- $f''(x) < 0$ when $x < 2$ and when $4 < x < 7$
- $f''(x) > 0$ when $2 < x < 4$ and when $x > 7$
- $f(0) = -4$

10. Find constants a, b, and c so that the function $f(x) = ax^3 + bx^2 + c$ will have a relative extremum at (2, 11) and a point of inflection at (1, 5). Sketch the graph of $y = f(x)$.

Part C

11. Find the value of the constant b so that the function $f(x) = \sqrt{x + 1} + \frac{b}{x}$ has a point of inflection at $x = 3$.

Thinking/Inquiry/ Problem Solving

12. Show that the graph of $f(x) = ax^4 + bx^3$ has two points of inflection. Show that the x-coordinate of one of these points lies midway between the x-intercepts.

13. a. Use the algorithm for curve sketching to sketch the function $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4}$.

b. Explain why it is difficult to determine the oblique asymptote using a graphing calculator.

Section 9.5 — An Algorithm for Graph Sketching

You now have the necessary skills to sketch the graphs of most elementary functions. However, you might be wondering why you should spend time developing techniques for sketching graphs when you have a graphing calculator. The answer is that, in doing so, you develop an understanding of the qualitative features of the functions you are analyzing. In this section, you will combine the skills you have developed. Some of them use the calculus properties. Others were learned earlier. Putting them all together allows you to develop an approach that leads to simple, yet accurate, sketches of the graphs of functions.

The algorithm for curve sketching now reads:

An Algorithm for Sketching the Graph of $y = f(x)$

Note: As each piece of information is obtained, make use of it in building the sketch.

Step 1: Determine any discontinuities or limitations in the domain. For discontinuities, investigate function values on either side of the discontinuity.

Step 2: Determine any vertical asymptotes.

Step 3: Determine any intercepts.

Step 4: Determine any critical points using $\frac{dy}{dx} = 0$.

Step 5: Test critical points to see whether they are local maxima, local minima, or neither.

Step 6: Determine the behaviour of the function for large positive and large negative values of x. This will identify horizontal asymptotes if they exist.

Step 7: Test for points of inflection.

Step 8: Determine any oblique asymptotes.

Step 9: Complete the sketch.

In using this algorithm, keep two things in mind:

1. You will not use all steps in every situation. Use only those that are essential.
2. You are familiar with the basic shapes of many functions. Use this knowledge when possible.

INVESTIGATION Use the algorithm for curve sketching to sketch the graphs of $y = -3x^3 - 2x^2 + 5x$ and $y = x^4 - 3x^2 + 2x$.

EXAMPLE 1 Sketch the graph of $f(x) = \frac{x - 4}{x^2 - x - 2}$.

Solution

The function is not defined if $x^2 - x - 2 = 0$.

$$\text{or } (x - 2)(x + 1) = 0$$
$$\text{or } x = 2 \text{ or } x = -1.$$

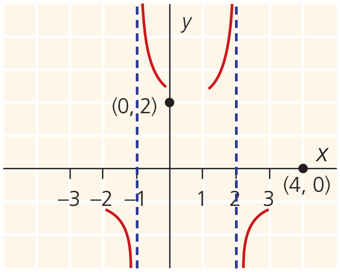
There are vertical asymptotes at $x = 2$ and $x = -1$.

Using $f(x) = \frac{x - 4}{(x - 2)(x + 1)}$, we examine function values near the asymptotes.

$$\lim_{x \to -1^-} f(x) = -\infty \qquad \lim_{x \to -1^+} f(x) = +\infty$$
$$\lim_{x \to 2^-} f(x) = +\infty \qquad \lim_{x \to 2^+} f(x) = -\infty$$

The x-intercept is 4 and the y-intercept is 2.

Sketch the information you have to this point, as shown.
Now determine the critical points.



$$f'(x) = \frac{(1)(x^2 - x - 2) - (x - 4)(2x - 1)}{}$$
$$= \frac{-x^2 + 8x - 6}{(x^2 - x - 2)^2}$$

$$f'(x) = 0 \quad \text{if} \quad -x^2 + 8x - 6 = 0$$
$$x = \frac{8 \pm 2\sqrt{10}}{2}$$
$$= 4 \pm \sqrt{10}$$

Since we are sketching, approximate values 7.2 and 0.8 are acceptable. These values give the points (7.2, 0.1) and (0.8, 1.5).

From the information we already have, we can see that (7.2, 0.1) is probably a local maximum and (0.8, 1.5) is probably a local minimum. Using the second derivative test to verify this is a difficult computational task. Instead, we can verify using the first derivative test, as in the following chart:

x	0	0.8	1	7	7.2	8
$f'(x)$	< 0	0	> 1	> 0	0	< 0

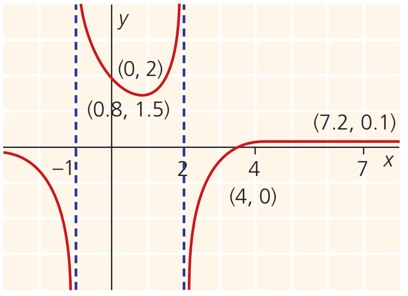
$x = 0.8$ gives the local minimum. $x = 7.2$ gives the local maximum.

Now check for large values of x.

$$\lim_{x \to \infty} \frac{x - 4}{x^2 - x - 2} = 0, \text{ but } y > 0 \text{ always;}$$
$$\lim_{x \to -\infty} \frac{x - 4}{x^2 - x - 2} = 0, \text{ but } y < 0 \text{ always.}$$

Therefore, $y = 0$ is a horizontal asymptote. The curve approaches $y = 0$ from above on the right and below on the left.

There is a point of inflection beyond $x = 7.2$ since the curve opens down at that point but changes as x becomes very large. The amount of work necessary to determine the point is greater than the information we gain, so we leave it undone here. (If you wish to check it, it occurs for $x \cong 10.4$). The finished sketch is given below, and because it is a sketch, it is not to scale.



EXAMPLE 2 Sketch the graph of $y = \dfrac{x^2 - 2}{x}$, showing all asymptotes of the curve.

Solution

Since y is undefined for $x = 0$, there is a vertical asymptote at $x = 0$.

Checking values near $x = 0$, we rewrite the function as $y = x - \dfrac{2}{x}$.

$$\lim_{x\to 0^-}\left(x - \frac{2}{x}\right) = +\infty \qquad \lim_{x\to 0^+}\left(x - \frac{2}{x}\right) = -\infty$$

If $y = 0$, $x^2 - 2 = 0$ and $x = \pm\sqrt{2}$, so the x-intercepts are $\sqrt{2}$ and $-\sqrt{2}$. There is no y-intercept.

Now determine the critical points.

$$\frac{dy}{dx} = 1 + \frac{2}{x^2} = \frac{x^2 + 2}{x^2}$$

$$\frac{dy}{dx} = 0 \quad \text{if} \quad x^2 + 2 = 0.$$

There are no real solutions of x, and therefore no critical values.

For large values of x, we examine $y = x - \dfrac{2}{x}$.

$$\lim_{x\to\infty}\frac{2}{x} = 0 \quad \text{but} \quad \frac{2}{x} > 0.$$

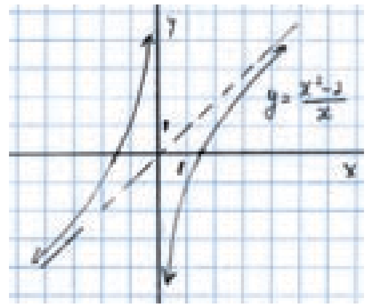
Hence $y = x - \dfrac{2}{x}$ approaches $y = x$ but is below the line.

$$\lim_{x\to -\infty}\frac{2}{x} = 0 \quad \text{but} \quad \frac{2}{x} < 0.$$

Hence $y = x - \dfrac{2}{x}$ approaches $y = x$ but is above the line.

Therefore, $y = x$ is an oblique asymptote.

The sketch can now be completed as shown.



Exercise 9.5

Part B

Knowledge/ Understanding

1. Use the algorithm for curve sketching to sketch the following:

a. $y = x^3 - 9x^2 + 15x + 30$

b. $f(x) = -4x^3 + 18x^2 + 3$

c. $y = 3 + \frac{1}{(x+2)^2}$

d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

e. $y = \frac{2x}{x^2 - 25}$

f. $y = \frac{1}{} e^{-\frac{x^2}{2}}$

g. $y = \frac{6x^2 - 2}{x^3}$

h. $s = \frac{50}{1 + 5e^{-0.01t}}, t \geq 0$

i. $y = \frac{x+3}{x^2 - 4}$

j. $y = \frac{x^2 - 3x + 6}{x - 1}$

k. $c = te^{-t} + 5$

l. $y = x(\ln x)^3$

Application

2. Determine the constants a, b, c, and d so that the curve defined by $y = ax^3 + bx^2 + cx + d$ has a local maximum at the point (2, 4) and a point of inflection at the origin. Sketch the curve.

3. Sketch the function defined by $g(x) = \frac{8e^x}{e^{2x} + 4}$.

4. Sketch the graph of $y = e^x + \frac{1}{x}$.

Part C

5. Sketch the graph of $f(x) = \frac{k - x}{k^2 + x^2}$, where k is any positive constant.

6. Sketch the curve defined by $g(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$.

7. Find the horizontal asymptotes for each of the following:

a. $f(x) = \frac{x}{}$

b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

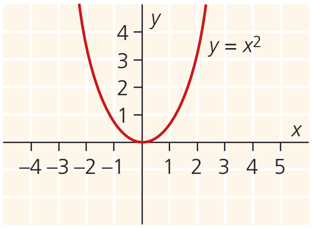
Thinking/Inquiry/ Problem Solving

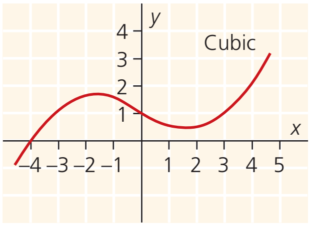
8. Show that for any cubic function of the form $y = ax^3 + bx^2 + cx + d$, there is a single point of inflection where the slope of the curve at that point is $c - \frac{b^2}{3a}$.

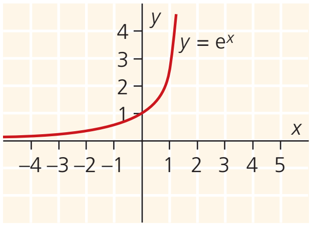
Key Concepts Review

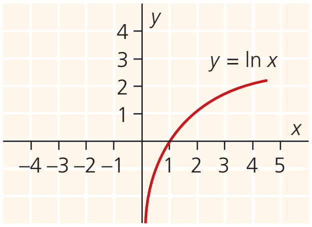
In this chapter, you saw that calculus can aid in sketching graphs. Remember that things learned in earlier studies are useful and that calculus techniques help in sketching. Basic shapes should always be kept in mind. Use these together with the algorithm for curve sketching, and always use accumulated knowledge.

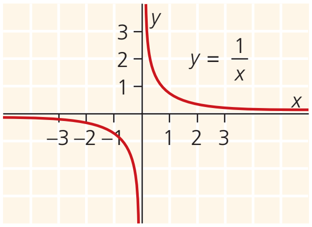
Basic Shapes to Remember

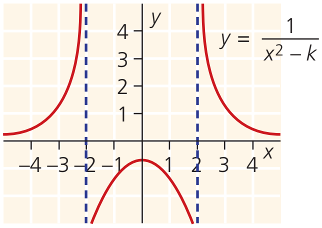












Career Link wrap-up

investigate and apply

CHAPTER 9: PREDICTING STOCK VALUES

In the Career Link earlier in the chapter, you investigated a graphical model used to predict stock values for a new stock. A brand new stock is also called an Initial Public Offering, or IPO. Remember that in this model, the period immediately after the stock is issued offers excess returns on the stock (i.e., the stock is selling for more than it is really worth). One such model for a class of Internet IPOs predicts the percentage overvaluation of a stock as a function of time, as

$$R(t) = 250\frac{t^2}{e^{3t}},$$

where $R(t)$ is the overvaluation in percent and t is the time in months after issue.

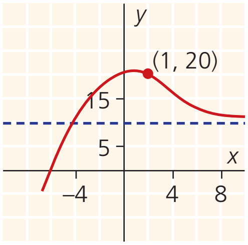
Use the information provided by the first derivative, second derivative, and asymptotes to prepare advice for clients as to when they should expect a signal to prepare to buy or sell (inflection point), the exact time when they should buy or sell (max/min), and any false signals prior to an asymptote. Explain your reasoning. Make a sketch of the function without using a graphing calculator.

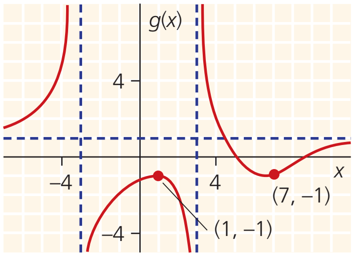
Review Exercise

1. Determine the derivative and the second derivative for each of the following:

   a. $y = e^{nx}$
   b. $f(x) = \ln(x + 4)^{\frac{1}{2}}$
   c. $s = \dfrac{e^t - 1}{e^t + 1}$
   d. $g(t) = \ln(t + \sqrt{1 + t^2})$

2. For each of the following graphs, state:
   i) the intervals where the function is increasing,
   ii) the intervals where the function is decreasing,
   iii) the points where the tangent to the function is horizontal.

   a.


   b.


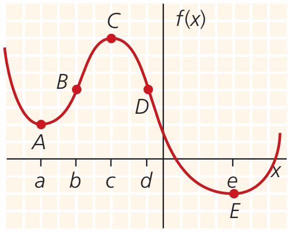
3. Is it always true that an increasing function is concave up in shape? Explain.

4. Find the critical points for each of the following. Determine whether the critical point is a local maximum or local minimum and whether or not the tangent is parallel to the horizontal axis.

   a. $f(x) = -2x^3 + 9x^2 + 20$
   b. $g(t) = \dfrac{e^{-2t}}{t^2}$
   c. $h(x) = \dfrac{x - 3}{x^2 + 7}$
   d. $k(x) = \ln(x^3 - 3x^2 - 9x)$

5. The graph for the function $y = f(x)$ has relative extrema at points A, C, and E and points of inflection at B and D. If a, b, c, d, and e are the x-coordinates for the points, then state the intervals in which each of the following conditions are true:

   a. $f'(x) > 0$ and $f''(x) > 0$
   b. $f'(x) > 0$ and $f''(x) < 0$
   c. $f'(x) < 0$ and $f''(x) > 0$
   d. $f'(x) < 0$ and $f''(x) < 0$



6. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

   a. $y = \frac{2x}{x-3}$
   b. $g(x) = \frac{x-5}{x+5}$
   c. $s = \frac{5}{2e^x - 8}$
   d. $f(x) = \frac{x^2 - 2x - 15}{x+3}$

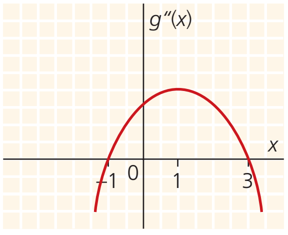
7. Determine the points of inflection for each of the following:

   a. $f(w) = \frac{\ln w^2}{w}$
   b. $g(t) = te^t$

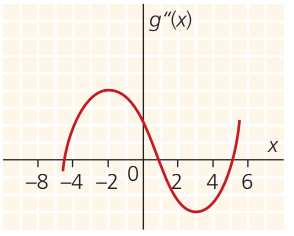
8. Sketch a graph of a function that is differentiable on the interval $-3 \le x \le 5$ and satisfies the following conditions:

   - local maxima at $(-2, 10)$ and $(3, 4)$,
   - the function f is decreasing on the intervals $-2 < x < 1$ and $3 \le x \le 5$,
   - the derivative $f'(x)$ is positive for $-3 \le x < -2$ and for $1 < x < 3$,
   - $f(1) = -6$.

9. Each of the graphs below represents the second derivative $g''(x)$ of a function $g(x)$.

   i)


   $g''(x)$ is a quadratic function

   ii)


   $g''(x)$ is a cubic function

   a. On what intervals is the graph of $g(x)$ concave up and on what intervals is the graph concave down?
   b. List the x-coordinates of the points of inflection.
   c. Make a rough sketch of a possible graph for $g(x)$, assuming that $g(0) = -3$.

10. a. If the graph of the function $g(x) = \frac{ax + b}{(x-1)(x-4)}$ has a horizontal tangent at point $(2, -1)$, then determine the values of a and b.
    b. Sketch the function g.

11. Find the equation of the oblique asymptote in the form $y = mx + b$ for each of the following, and then show that $\lim_{x \to +\infty} [y - f(x)] = 0$ for each of the functions given.

a. $f(x) = \frac{2x^2 - 7x + 5}{}$ b. $f(x) = \frac{4x^3 - x^2 - 15x - 50}{}$

12. Sketch the following using suitable techniques.

a. $y = x^4 - 8x^2 + 7$

b. $f(x) = \frac{3x - 1}{x + 1}$

c. $g(x) = \frac{x^2 + 1}{4x^2 - 9}$

d. $y = 3x^2 \ln x$

e. $h(x) = \frac{x}{x^2 - 4x + 4}$

f. $f(t) = \frac{t^2 - 3t + 2}{t - 3}$

g. $s = te^{-3t} + 10$

h. $P = \frac{100}{1 + 50e^{-0.2t}}$

13. The population, P, of a laboratory colony of bacteria is given by the formula $P = 10^4 te^{-0.2t} + 100$ where t is the time in days since the creation of the colony.

a. Find the maximum number of bacteria the colony will sustain and when this maximum is reached.

b. Find the time when the rate of change of the growth rate of the colony starts to increase.

c. Sketch the curve for the first 15 days.

14. Prove the second derivative of the function $y = \ln\left[\frac{x^2 + 1}{x^2 - 1}\right]$ is positive for all real values of x except $|x| \leq 1$.

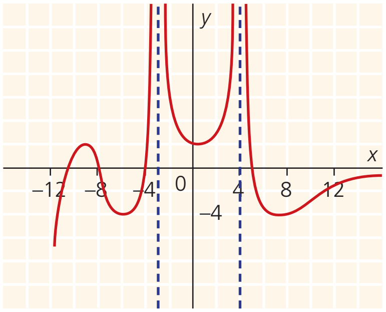
15. a. Find the conditions on parameter k so that the function $f(x) = \frac{2x + 4}{x^2 - k^2}$ will have critical points.

b. Select a value for k that satisfies the constraint established in part **a** and sketch the section of the curve that lies in the domain $|x| \leq k$.

Chapter 9 Test

Achievement Category	Questions
Knowledge/Understanding	1, 2, 4, 8, 10
Thinking/Inquiry/Problem Solving	2, 3, 5, 7, 8, 10
Communication	1, 9
Application	3, 6, 9

1. A function $y = f(x)$ is defined in the following graph.
   a. State the intervals where the function is increasing.
   b. State the intervals where $f'(x) < 0$.
   c. Write the coordinates for the critical points.
   d. Write the equations for any vertical asymptotes.
   e. What is the value of $f''(x)$ on the interval $-3 < x < 3$?
   f. If $x \geq -6$, state the intervals where $f'(x) < 0$ and $f''(x) > 0$.
   g. Identify a point of inflection and state the approximate ordered pair for the point.



2. a. Find the critical points for the function $g(x) = 2x^4 - 8x^3 - x^2 + 6x$.
   b. Determine the type of each critical point in part **a.**

3. Sketch the graph of a function with the following properties:
   - There are relative extrema at $(-1, 7)$ and $(3, 2)$.
   - There is a point of inflection at $(1, 4)$.

- The graph is concave down only when $x < 1$.
- The x-intercept is -4 and the y-intercept is 6.

4. Check the function $g(x) = \frac{x^2 + 7x + 10}{(x - 3)(x + 2)}$ for discontinuities. Conduct appropriate tests to determine if asymptotes exist at the discontinuity values. State the equations of any asymptotes.

5. Find the critical points for the function $g(x) = e^{2x}(x^2 - 2)$, and determine the type of critical point by using an appropriate test.

6. Use at least five curve-sketching techniques to explain how to sketch the graph of the function $f(x) = \frac{2x + 10}{x^2 - 9}$. Sketch the graph on squared paper.

7. The function $y = kx^2 + \ln(kx)$ has $\frac{d^2y}{dx^2} = -\frac{33}{16}$ when $x = \frac{4}{11}$. Determine the value of k.

8. The function $f(x) = x^3 + bx^2 + c$ has a critical point at $(-2, 6)$.
   a. Find the constants b and c.
   b. Sketch the graph of $f(x)$ using only the critical points and the second derivative test.

9. Use appropriate techniques to sketch the function $y = x^{\frac{2}{3}}(x - 5)$. Explain your work.

10. For the function $y = x^2 e^{kx} + p$, the slope of the tangent is zero when $x = \frac{2}{3}$ (k and p are parameters).
    a. Determine the value of k.
    b. Describe the role of p in this function.

Cumulative Review

CHAPTERS 3–9

1. Write the first four terms for the sequences defined by the given function and then find the limit for term t_n in the sequence as $n \to \infty$.

   a. $f(n) = 2 - \frac{1}{5^n}$
   b. $g(k) = \frac{1}{k(k+1)}$

2. Evaluate each of the following limits:

   a. $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 5x + 4}$
   b. $\lim_{x \to 0} \frac{x}{2x - x^2}$
   c. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$
   d. $\lim_{x \to \infty} \frac{5 - 2x^2}{3x + 5x^2}$
   e. $\lim_{x \to \infty} \frac{2^{\frac{1}{n}} - 3^{-n}}{2^n}$
   f. $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{\sqrt{2}x}$
   g. $\lim_{t \to 0} \frac{4t^2 + 3t + 2}{t^3 + 2t - 6}$
   h. $\lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$
   i. $\lim_{h \to 0} \frac{x^2h + 3xh^2 + h^3}{2xh + 5h^2}$

3. If $f(x) = x^3 - 5x^2 + 10x$, find $\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

4. Use the method of first principles to find the derivative function of each of the following:

   a. $s = t^2 + 10t$
   b. $y = \frac{2 - x}{x^2}$

5. Use either the Product Rule or the Quotient Rule to find the derivative of each of the following:

   a. $s = (t^2 - 5t)(5t^3 - 2t + 7)$
   b. $y = \frac{x^3 - 3x}{x^2 + 2x + 5}$
   c. $v = e^w(1 + w)$
   d. $s = \frac{e^t - e^{-t}}{e^t + e^{-t}}$
   e. $y = (e^x)(\ln x)$
   f. $s = (\ln t + e^t)t$

6. Use the Chain Rule to find the derivative of each of the following:

   a. $s(t) = e^{t^2 - 5t}$
   b. $y = \ln(x^2 + x + 1)$
   c. $w = \sqrt{3x + \frac{1}{x}}$
   d. $g(t) = 3e^{(2t + \ln t)}$
   e. $p = \log_a(2r) + 3\ln(5r)$
   f. $e^{x+y} = xy$
   g. $y = (a^2 - x^2)^{-\frac{1}{2}}$
   h. $\ln(x^2y) = 2y$

7. Find the slope of the tangent to the circle defined by $x^2 + y^2 = 100$ at the point $(-6, -8)$.

8. Use the appropriate derivative rule to find the derivative of each of the following. In all cases, the values a, b, and c are constants.

   a. $w = r(2^r) - r^2e^{2r}$

   b. $z = w\sqrt{a + bw}$

   c. $s = 3\sqrt{\dfrac{2 + 3t}{2 - 3t}}$

   d. $y = e^x - 2e^{-x}$

   e. Find $\dfrac{dy}{dx}$ for $b^2x^2 + a^2y^2 = a^2b^2$.

   f. Find $\dfrac{dy}{dx}$ for $x^3 + 3x^2y + y^3 = c^3$.

9. Find the equation of the tangent to the curve defined by $s = te^{t^2}$ at the point where $t = \pi$.

10. Find the value of k in the equation $y = e^{kx}$ so that y is a solution of each of the following:

    a. $y'' - 3y' + 2y = 0$

    b. $y''' - y'' - 4y' + 4y = 0$

11. Use implicit differentiation to find the second derivative of the relation $x^2 + 6xy - y^2 = 10$.

12. Find the slope of the tangents to the curve defined by $y^2 = e^{2x} + 2y - e$ at the point where $y = 2$.

13. Find the equation of any tangent to the curve represented by $x^2 - xy + 3y^2 = 132$ that is parallel to the straight line defined by $x - y = 2$.

14. Find the equations of the straight lines through point $A(3, -2)$ that are tangent to the curve defined by $y = x^2 - 7$.

15. Find the equation of the tangent to the curve defined by $y = x + \ln x$ that is perpendicular to the line defined by $3x + 9y = 8$.

16. A parachutist jumps out of an airplane. The distance, s, (in metres), through which she falls in t seconds is given by $s(t) = 10t - \dfrac{6t}{t + 1}$. Determine

    a. the distance through which she falls in the first second.

    b. the velocity of the parachutist at $t = 1$ and $t = 2$.

    c. the acceleration of the parachutist at $t = 1$ and $t = 2$.

    d. Find the limit of the velocity as $t \to \infty$. This limit is known as the "terminal velocity."

17. As a particle travels in a linear direction, the distance s from the origin is given by $s = 8 - 7t + t^2$, where t is in seconds and s is in millimetres.

    a. Find the velocity after 3 s.

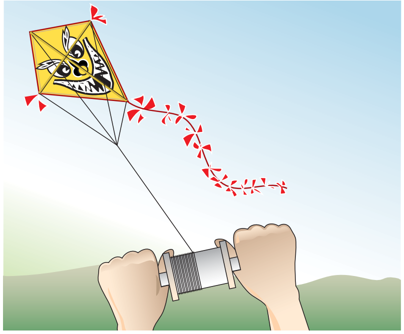
b. Find the average velocity in the fourth second.

c. Find the acceleration after 3 s.

18. a. The radius of spherical soap bubble is expanding at the rate of 2 mm/s. At what rate is the surface of the soap bubble increasing when the radius is 7 mm?

    b. Repeat part **a** for the situation where the rate of expansion is 1 mm/s.

19. A kite 50 m high is being blown parallel to the ground at a rate of 3 m/s. The person flying the kite is standing still. How fast is the kite string running out at the instant when exactly 100 m of string are out?



20. A conical cistern 5 m deep and 8 m across the top is being filled with water flowing in at a rate of 10 000 cm^3/min.

    a. Explain the significance of the derivatives $\frac{dv}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$.

    b. Use the geometry of the configuration to find a formula for the volume of the water in terms of the radius of the surface of the water, r.

    c. At what rate is the water rising in the cistern when the depth is 3 m?

21. The equation of motion of a particle moving in a straight line is $s = kv^2 \ln v$, where k is a constant and v is the velocity. Find an equation that expresses the acceleration in terms of velocity.

22. A car leaves a small town at 13:00 and travels due south at a speed of 80 km/h. Another car has been heading due west at 100 km/h and reaches the same town at 15:00. At what time were the two cars closest together?

23. Find the local extreme points and the points of inflection for each of the following:

    a. $y = 2x^3 + 3x^2 - 36x + 10$

    b. $w = 4 - \frac{100}{z^2 + 25}$

    c. $f(x) = x^2 \ln x$

    d. $y = x^3 e^{-2x}$

    e. $y = 5xe^{-\frac{x^2}{4}}$

    f. $n = 10pe^{-p} + 2$

24. For each of the following, determine the equations of any horizontal, vertical, or oblique asymptotes and find any local extremes.

    a. $y = \frac{8}{x^2 - 9}$

    b. $y = \frac{4x^3}{x^2 - 1}$

25. Use the curve-sketching techniques that you think are appropriate to sketch each of the following:

a. $p = \dfrac{10n^2}{n^2 + 25}$ b. $y = x \ln(3x)$

c. $y = \dfrac{3x}{x^2 - 4}$ d. $y = 10^{-\frac{x^2}{4}}$

26. A farmer has 750 m of fencing and wants to enclose a rectangular area on all four sides, then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible area of the four pens?

27. A metal can is made to hold 500 mL of soup. Find the dimensions of the can that will minimize the amount of metal required. (Assume that the top and sides of the can are made from metal of the same thickness.)

28. A cylindrical box of volume 4000 cm^3 is being constructed to hold Christmas candies. The cost of the base and lid is \$0.005/cm^2 and the cost of the side walls is \$0.0025/cm^2. Find the dimensions for the cheapest possible box.

29. An open rectangular box has a square base with each side x cm.

a. If the length, width, and depth have a sum of 140 cm, find the depth.

b. Find the maximum possible volume you could have when constructing a box with these proportions, and find the dimensions to make this maximum volume.

30. The price of x items of a certain type of product is $p(x) = 50 - x^2$, where $x \in N$. If the total revenue $R(x)$ is given by $R(x) = xp(x)$, find the value of x that corresponds to the maximum possible total revenue.

31. A fish biologist introduced a new species of fish into a northern lake and studied the growth of the population over a period of ten years. The mathematical model that best described the size of the fish population was $p = \dfrac{4000}{1 + 3e^{-0.1373t}}$, where t is in years.

a. Find the maximum population that the biologist expects in the lake.

b. Find the year when the rate of change of the growth rate started to decrease.

c. Sketch the curve for the ten-year period.

d. For how many more years must the biologist collect data to be sure the mathematical model is valid?

32. Determine values for a, b, c, and d that guarantee that the function $f(x) = ax^3 + bx^2 + cx + d$ will have a relative maximum at $(1, -7)$ and a point of inflection at $(2, -11)$.

33. Sketch a graph of a function f that has the following properties:

- $f'(x) > 0$, when $x < 2$
- $f'(x) < 0$, when $x > 2$
- $f''(x) > 0$, when $x < 2$
- $f''(x) < 0$, when $x > 2$

34. Determine the extreme values of each function on the given interval.

a. $f(x) = 1 + (x + 3)^2$, $-2 \le x \le 6$

b. $f(x) = x + \frac{1}{\sqrt{x}}$, $1 \le x \le 9$

c. $f(x) = \frac{e^x}{1 + e^x}$, $0 \le x \le 4$

d. $f(x) = x + \ln(x)$, $1 \le x \le 5$

35. A travel agent booking a tour currently has 80 people signed up. The price of a ticket is \$5000 per person. The agency has chartered a plane seating 150 people at a cost of \$250 000. Additional costs to the agency are incidental fees of \$300 per person. For each \$30 that the price is lowered, one new person will sign up. How much should the price per person be lowered in order to maximize the profit to the travel agent?

36. Find the equations of the tangents to the curve defined by $x^2 + xy + y^2 = 19$ at the points on the curve where $y = 2$.

37. Use the techniques of curve sketching that you think are appropriate to sketch the curve defined by $y = \frac{4}{x^2 - 4}$.

38. The Coast Guard is monitoring a giant iceberg in the approximate shape of a rectangular solid that is five times as long as the width across the front face. As the iceberg drifts south, the height above water is observed to decrease at the rate of two metres per week, and the width across the front is shrinking at three metres per week. Find the rate of loss of volume above water when the height is 60 m and the width of the face is 300 m.

39. Determine values for a, b, c, and d that guarantee that the function $f(x) = ax^3 + bx^2 + cx + d$ has a relative maximum value of 3 when $x = -2$ and a relative minimum value of 0 when $x = 1$.

40. Find the equation of the normal to each of the curves defined below, at the point specified.

a. $y = x^3 + 2x^2 + 5x + 2$, when $x = -1$

b. $y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$ at $(4, 2.5)$